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Quantum double structure for Heisenberg–Weyl algebra without q-deformation

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Abstract. In this paper, a new quasi-triangular Hopf algebra is presented as the quantum double of the Heisenberg-Weyl algebra without q-deformation. Its universal R-matrix is built and the corresponding representation theory is studied with the explicit construction for the representations of this quantum double.

1. Introduction

Recently, the quantum group theory associated with the Yang–Baxter equation for a nonlinear integrable system has became the focus of attention from both theoretical physicists and mathematicians [1]. As a kernel of this theory, Drinfeld's quantum double construction is a quite powerful tool in constructing the solutions, namely the R-matrices, for the quantum Yang–Baxter equation (QYBE) in connection with certain algebraic structures, such as quantum algebras [2], quantum super algebras [3], quantum affine algebras [2], their multiparameter deformations [4] and the quantum doubles of the Borel subalgebras for universal enveloping algebras (UEA) of classical Lie algebra [6]. Some of them usually are so-called q-deformations and similar constructions are studied by different authors, for example, in [7].

In this paper we will present a different quasi-triangular Hopf algebra that is the quantum double of the Heisenberg–Weyl(HW) algebra based on Drinfeld's quantum double construction. To construct the explicit R-matrices for the QYBE from its universal R-matrix of this quantum double, we study its representation theory and explicitly construct its finite and infinite dimensional representations. A six-dimensional example of R-matrices for this quantum double is given as an illustration. The studies in this paper shows that, like its q-deformation [6–8], the ordinary HW algebra also realizes a so-called quantum group structure, quasi-triangular Hopf algebra associated QYBE. This fact shows that the canonical quantization defined by the HW algebra possibly prompts an important role of 'quantum group structure' and the QYBE in quantum theory.

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2. Quantum double of HW algebra

The Heisenberg–Weyl algebra (HW) algebra A is an associative algebra generated by a, $\bar{a}$, E and the unit 1. These generators satisfy the defining relations

$$[a, \bar{a}]=E, [E, a]=0=[E, \bar{a}]. \tag{2.1}$$

If we take a special representation T such that

$$T(a)^{+}=T(\bar{a}), \ T(E)=\text{unit matrix } I$$

then $\bar{a}$ and a can be regarded as the creation and annihilation operators of boson states in second quantization. Since the algebra A is the UEA of the HW Lie algebra with basis $\{a, \bar{a}, E\}$, A can be endowed with a well known Hopf algebraic structure

$$\Delta(x)=x\otimes 1+1\otimes x, \ S(x)=-x, \ \varepsilon(x)=0 \tag{2.2}$$

for $x=a, \bar{a}, E$ where the algebraic homomorphisms Δ, ε and the algebraic antihomomorphism S defined only for the generators are naturally extended for the whole algebra. According to the PBW theorem, the basis for the algebra A is chosen as

$$\{X(m, n, s)=\bar{a}^m a^n E^s, m, n, s, \in \boldsymbol{Z}^{+}=\{0, 1, 2, \ldots\}\}.$$

Now, let us consider the dual Hopf algebra $\boldsymbol{B}$ of A. Suppose $\bar{b}$, b and H are the dual generators to $\bar{a}$, a and E respectively and then defined by

$$\begin{aligned} &\langle X(m, n, s), \bar{b}\rangle=\delta_{m,1}\delta_{n,0}\delta_{s,0} \\ &\langle X(m, n, s), b\rangle=\delta_{m,0}\delta_{n,1}\delta_{s,0} \\ &\langle X(m, n, s), E\rangle=\delta_{m,0}\delta_{n,0}\delta_{s,1}. \end{aligned} \tag{2.3}$$

Since the algebra A is commutative, its Hopf algebraic dual is Abelian, i.e. the dual generators commute each other. Choosing a basis for $\boldsymbol{B}$

$$Y(m, n, s)=(m!\, n!\, s!)^{-1}\bar{b}^m b^n H^s, m, n, s, \in \boldsymbol{Z}^{+} \tag{2.4}$$

we prove the following proposition:

Proposition 1. The equations (2.4) define a dual basis $Y(m, n, s)$ for $\boldsymbol{B}$ satisfying

$$\langle X(m, n, s), Y(m', n', s')\rangle=\delta_{m,m'}\delta_{n,n'}\delta_{s,s'}. \tag{2.5}$$

Proof. According to the Hopf algebraic duality between A and $\boldsymbol{B}$:

$$\begin{aligned} &\langle a, b_1 b_2\rangle=\langle\Delta_A(a), b_1\otimes b_2\rangle, a\in \boldsymbol{A}, b_1, b_2\in \boldsymbol{B} \\ &\langle a_1 a_2, b\rangle=\langle a_2\otimes a_1, \Delta_B(b)\rangle, a_1, a_2\in \boldsymbol{A}, b\in \boldsymbol{B} \\ &\langle 1_A, b\rangle=\varepsilon_B(b), b\in \boldsymbol{B} \\ &\langle a, 1_B\rangle=\varepsilon_A(a), a\in \boldsymbol{A} \\ &\langle S_A(a), S_B(b)\rangle=\langle a, b\rangle, a\in \boldsymbol{A}, b\in \boldsymbol{B} \end{aligned} \tag{2.6}$$

where for $C=\boldsymbol{A}, \boldsymbol{B}$, the operations Δ_C, ε_C and S_C are the coproduct, co-unit and antipode of C, respectively; 1_C is the unit of C. Without confusion we no longer use the index C to specify Δ_C, ε_C and S_C. Let $G=\bar{a}, a, E$ corresponding to $F=\bar{b}, b, H$, respectively. For $G\neq\bar{a}$

$$\begin{aligned}\langle\bar{a}^s G^m, F^n\rangle&=\langle\Delta(\bar{a}^s G^m), F^{n-1}\otimes F\rangle\\&=\sum_{k,l=0}^{m,s}\frac{m!\,s!}{(m-k)!\,k!\,(s-l)!\,l!}\langle\bar{a}^{s-l}G^{m-k}\otimes\bar{a}^l G^k, F^{n-1}\otimes F\rangle\\&=m\langle\bar{a}^s G^{m-1}, F^{n-1}\rangle=m!\,\delta_{m,n}\delta_{s,0}\end{aligned}$$

where we have used

$$\langle G^m, F^n\rangle=m!\,\delta_{m,n}$$

which follows from

$$\langle G^m, F^n\rangle=\langle\Delta(G^m), F^{n-1}\otimes F\rangle.$$

For $F\neq\bar{b}$, similarly, we have

$$\langle G^m, \bar{b}^s F^n\rangle=m!\,\delta_{m,n}\delta_{s,0}.$$

Then, we have

$$\langle\bar{a}^m a^n, \bar{b}^s b^l\rangle=\langle\Delta(\bar{a}^m a^n), \bar{b}^s\otimes b^l\rangle=s!\,l!\,\delta_{m,s}\delta_{n,l}$$

and therefore prove equation (2.5).

It follows from the above proposition that

$$\begin{aligned}\langle X(m,n,s)\otimes X(k,l,r), \Delta(H)\rangle&=\langle X(k,l,r)X(m,n,s), H\rangle\\&=\delta_{m,0}\delta_{n,0}\delta_{k,0}\delta_{l,0}(\delta_{s,1}\delta_{r,0}+\delta_{r,1}\delta_{s,0})+\delta_{m,1}\delta_{n,0}\delta_{s,0}\delta_{k,0}\delta_{l,1}\delta_{r,0}\end{aligned}$$

namely

$$\Delta(H)=H\otimes 1+1\otimes H+\bar{b}\otimes b.$$

Similarly, we calculate other operations of the generators for $\boldsymbol{B}$ under Δ, ε and S. The results are summarized as follows.

Proposition 2. The dual Hopf algebra $\boldsymbol{B}$ is generated by $\bar{b}$, b and H and endowed with the following Hopf algebraic structure

$$\begin{aligned}&\Delta(x)=x\otimes 1+1\otimes x\\&S(x)=-x,\ \varepsilon(x)=0,\ x=\bar{b}, b\\&\Delta(H)=H\otimes 1+1\otimes H+\bar{b}\otimes b.\end{aligned}\tag{2.7}$$

3. Quantum double and universal R-matrix

It should be noticed that the dual Hopf algebraic structure of $\boldsymbol{B}$ can also be obtained from the formal group theory [8] of Lie algebra in principle where the explicit expressions of the Baker–Comppell–Hausdorff formula for the HW Lie algebra. In this sense, Drinfeld's theory is not a unique approach to obtain the dual Hopf algebraic structure. However, it is important that Drinfeld's theory can also provide us with a convenient method to 'combine' $\boldsymbol{A}$ and $\boldsymbol{B}$ to form a 'larger' Hopf algebra $\boldsymbol{D}$ containing $\boldsymbol{A}$ and $\boldsymbol{B}$

as subalgebras. The universal R-matrix for QYBE can be automatically given in this construction.

According to the multiplication formula for the quantum double

$$ba=\sum_{i,j}\langle a_i(1), S(b_j(1))\rangle\langle a_i(3), b_j(3)\rangle a_i(2)b_j(2) \tag{3.1}$$

where $c_i(k)$ $(k=1, 2, 3; c=a, b)$ are defined by

$$\Delta^2(c)=(\mathrm{id}\otimes\Delta)\Delta(c)=(\Delta\otimes\mathrm{id})\Delta(c)=\sum_i c_i(1)\otimes c_i(2)\otimes c_i(3).$$

Using the explicit expressions

$$\Delta^2(H)=H\otimes 1\otimes 1+1\otimes H\otimes 1+1\otimes 1\otimes H+1\otimes\bar{b}\otimes b+\bar{b}\otimes b\otimes 1+\bar{b}\otimes 1\otimes b \tag{3.2}$$

we have:

Proposition 3. The quantum double $\boldsymbol{D}$ is generated by $a, \bar{a}, b, \bar{b}, E, H$ as an associative algebra with the only nonzero commutators

$$[a, \bar{a}]=E, [H, a]=\bar{b}, [H, \bar{a}]=-b \tag{3.3}$$

and as a non-cocommutative Hopf algebra with the structure (2.2) and (2.7). The universal R-matrix, a canonical element entwining $\boldsymbol{A}$ and $\boldsymbol{B}$, is

$$\hat{R}=\sum_{m,n,s=0}^{\infty} X(m, n, s)\otimes Y(m, n, s)=\exp(\bar{a}\otimes\bar{b})\exp(a\otimes b)\exp(E\otimes H). \tag{3.4}$$

Notice that the proof of the above proposition reduces essentially to proposition 1.

Using the above commutation relations, we can verify the following quasi-triangular relations directly

$$\begin{aligned}
&\hat{R}\Delta(x)=\sigma\Delta(x)\hat{R}\\
&(\Delta\otimes\mathrm{id})\hat{R}=\hat{R}_{13}\hat{R}_{23}\\
&(\mathrm{id}\otimes\Delta)\hat{R}=\hat{R}_{13}\hat{R}_{12}\\
&(\varepsilon\otimes\mathrm{id})\hat{R}=1=(\mathrm{id}\otimes\varepsilon)\hat{R}\\
&(S\otimes\mathrm{id})\hat{R}=\hat{R}^{-1}=(\mathrm{id}\otimes S)\hat{R}
\end{aligned} \tag{3.5}$$

where σ is such a permutation that $\sigma(x\otimes y)=y\otimes x$, $x, y\in \boldsymbol{D}$. The equations (3.5) mean that the above universal R-matrix satisfies the abstract QYBE

$$\hat{R}_{12}\hat{R}_{13}\hat{R}_{23}=\hat{R}_{23}\hat{R}_{13}\hat{R}_{12} \tag{3.6}$$

where $\hat{R}_{12}=\Sigma_m a_m\otimes b_m\otimes 1$, $\hat{R}_{13}=\Sigma_m a_m\otimes 1\otimes b_m$, $\hat{R}_{23}=\Sigma_m 1\otimes a_m\otimes b_m$ and a_m and b_m are the dual bases vectors of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. Here, we simply note $\hat{R}=\Sigma_m a_m\otimes b_m$.

4. On representations and realizations of the quantum double

In order to obtain the R-matrices for QYBE from the new universal R-matrix (3.5), we should consider the representations of the quantum double $\boldsymbol{D}$. For simplicity we denote $T(x)$ by x for a representation T of $\boldsymbol{D}$ sometimes as follows.

Proposition 4. There does not exist a finite-dimensional irreducible representation of $\boldsymbol{D}$ except the trivial representations T for which there at least is one generator s such that $T(x)=0$.

Remark that the one dimensional representation is trivial because the Abelian T is a scalar

$$T(E)=[T(a),\,T(\bar{a})]=0.$$

Proof. Thanks to the Schur lemma, we know that the representatives of the central elements E, $\bar{b}$ and b must be non-zero scalars times a unit matrix I for a n-dimensional irreducible representation, i.e.

$$E=I\eta\neq 0 \qquad b=I\xi\neq 0 \qquad \bar{b}=I\bar{\xi}\neq 0 \qquad \eta,\,\xi,\,\bar{\xi}\in\text{complex field } C.$$

However, taking the trace of E, we have

$$n\eta=\operatorname{tr}(E)=\operatorname{tr}([a,\bar{a}])=0$$

that is $\eta=0$. Then, a contradiction appears for the non-trivial representations.

From this proposition and its proof, we see that the finite-dimensional representation of $\boldsymbol{D}$ must be neither irreducible nor a sum of some non-trivially irreducible representations. The possible non-trivial finite dimensional representations are only those indecomposable ones, the reducible but not completely reducible representations where $\operatorname{tr}(E)=0$. For the former we can give a boson realization

$$\begin{gathered} a=c \qquad \bar{a}=c^{+} \qquad b=-\alpha\in C \qquad \bar{b}=-\beta\in C \\ E=1 \qquad H=\alpha c+\beta c^{+} \end{gathered} \tag{4.1}$$

in terms of the boson operators c and c^{+} satisfying

$$[c,c^{+}]=1 \qquad 1x=x1=x \qquad x=c,\,c^{+}. \tag{4.2}$$

Using the Fock representation of c and c^{+}

$$\begin{aligned} c^{+}|n\rangle&=(n+1)^{1/2}|n+1\rangle \\ c|n\rangle&=n^{1/2}|n-1\rangle \end{aligned} \tag{4.3}$$

on the Fock space

$$\{|n\rangle=(n!)^{-1/2}(c^{+})^{n}|0\rangle|c|0\rangle=0,\,n=0,1,2,\ldots\}$$

we obtain an infinite irreducible representation of $\boldsymbol{D}$ with explicit matrix elements

$$\begin{aligned} &(\bar{a})_{m,n}=(n+1)^{1/2}\delta_{m,n+1},\,(a)_{m,n}=n^{1/2}\delta_{m,n} \\ &(E)_{m,n}=\delta_{m,n},\,(b)_{m,n}=-\alpha\,\delta_{m,n},\,(\bar{b})_{m,n}=-\beta\,\delta_{m,n} \\ &(H)_{m,n}=\beta(n+1)^{1/2}\delta_{m,n+1}+\alpha n^{1/2}\delta_{m,n}\delta_{m,n-1}. \end{aligned} \tag{4.4}$$

In this realization and the corresponding representation, the universal R-matrix (3.4) can be expressed as a generator

$$R=\mathrm{e}^{-\beta C^{+}}\,\mathrm{e}^{-\alpha C}\otimes\mathrm{e}^{\beta C^{+}+\alpha C}=\mathrm{e}^{-\beta\alpha/2}D(-\beta,-\alpha)\otimes D(\beta,\alpha) \tag{4.5}$$

for the two-mode coherent state

$$|-\beta, \beta\rangle = N^{-1} \sum_{m,n=0}^{\infty} \frac{(-\beta c^{+})^{m} \otimes (\beta c^{+})^{n}}{m!\,n!} |0\rangle \tag{4.6}$$

where

$$D(\beta, \alpha) = e^{\beta C^{+} + \alpha C}$$

is a non-normalized single coherent state operator.

5. Explicit representations

Since the quantum Double $\boldsymbol{D} \simeq U(L)$ where $U(L)$ is the universal enveloping algebra of a Lie algebra L with the basis

$$\{a, \bar{a}, b, \bar{b}, H, E\}$$

we can construct many representations for the quantum double $\boldsymbol{D}$ as the subquotients of $L^{\otimes N}$. This means that the representation theory of $\boldsymbol{D}$ is enjoyed by the well known one for a nilpotent Lie algebra L. However, in order to obtain the finite dimensional R-matrices in *matrix form*, we must know the matrix representations of $\boldsymbol{D}$ explicitly. According to the above-mentioned general rule, we construct them as follows.

The PBW theorem determines the basis for $\boldsymbol{D}$

$$X[M] = X(m, n, l, r, s, t) = a^{m} \bar{a}^{n} H^{l} b^{r} \bar{b}^{s} E^{t}$$

where $m, n, l, r, s, t \in \boldsymbol{Z}^{+}$ and M denotes a six-vector $M = (m, n, l, r, s, t)$ in a lattice vector space $\boldsymbol{Z}^{+6}$ with the basis

$$\begin{aligned} e_1 &= (1, 0, 0, 0, 0, 0) & e_2 &= (0, 1, 0, 0, 0, 0) \\ e_3 &= (0, 0, 1, 0, 0, 0) & e_4 &= (0, 0, 0, 1, 0, 0) \\ e_5 &= (0, 0, 0, 0, 1, 0) & e_6 &= (0, 0, 0, 0, 0, 1). \end{aligned}$$

An explicit representation of $\boldsymbol{D}$ on the basis $X[M]$ is written explicitly as:

Proposition 5. The regular representation of $\boldsymbol{D}$ is

$$\begin{aligned} aX[M] &= X[M + e_1] \\ \bar{a}X[M] &= X[M + e_2] - mX[M - e_1 + e_6] \\ EX[M] &= X[M + e_6] \\ bX[M] &= X[M + e_4] \\ \bar{b}X[M] &= X[M + e_5] \\ HX[M] &= X[M + e_3] + mX[M - e_1 + e_5] - nX[M - e_2 + e_4]. \end{aligned} \tag{5.1}$$

Proof. Follows from the following equations

$$\begin{aligned}
&[\bar{a}, a^n] = -nEa^{n-1}\\
&[a, \bar{a}^n] = nE\bar{a}^{n-1}\\
&[H, a^n] = nE\bar{b}a^{n-1}\\
&[H, \bar{a}^n] = -nEb\bar{a}^{n-1}
\end{aligned} \tag{5.2}$$

which are obtained from (3.3) by induction.

Let I be a left ideal generated by the element $H-\mu$, i.e.

$$L(\mu) = \boldsymbol{D}(H-\mu) = \{x(H-\mu) | x \in \boldsymbol{D}\}.$$

Because the left ideal I is a left-invariant $\boldsymbol{D}$-submodule, on the quotient space $V(\mu) = \boldsymbol{D}/I(\mu)$:

$$u(K) = a^m \bar{a}^n b^r \bar{b}^s E^t \text{ Mod } I(\mu)$$

where $K = (m, n, r, s, t)$, $m, n, r, s, t \in \boldsymbol{Z}^+$, the regular representation induces an infinite-dimensional representation

$$\begin{aligned}
&au[K] = u[K+e_1]\\
&\bar{a}u[K] = u[K+e_2] - mu[K-e_1+e_5]\\
&Eu[K] = u[K+e_5]\\
&bu[K] = u[K+e_3]\\
&\bar{b}u[K] = u[K+e_4]\\
&Hu[K] = \mu u[K] + mu[K-e_1+e_4] - nu[K-e_2+e_3]
\end{aligned} \tag{5.3}$$

where

$$\begin{aligned}
&e_1 = (1, 0, 0, 0, 0) \qquad && e_2 = (0, 1, 0, 0, 0)\\
&e_3 = (0, 0, 1, 0, 0) \qquad && e_4 = (0, 0, 0, 1, 0)\\
&e_5 = (0, 0, 0, 0, 1).
\end{aligned}$$

Now, let us make a key observation from (5.3) that the sum $m+n+r+s+t$ for the basis vectors $u[K] = u(m, n, r, s, t)$ do not decrease under the actions of $\boldsymbol{D}$. This fact tells us that the following vectors

$$\{u[K] = u(m, n, r, s, t) | m+n+r+s+t \geqslant N\}$$

for a fixed $N \in \boldsymbol{Z}^+$ span an invariant subspace $V(\mu, N)$. Then, the quotient space $Q(\mu, N) = V(\mu)/V(\mu, N)$:

$$\text{Span}\{v(K) = u[K] \text{ Mod } V(\mu, N) | m+n+r+s+t \leqslant N-1\}$$

is finite dimensional and its dimension is

$$d(N) = \sum_{k=0}^{N-1} \frac{(k+4)!}{k!\,4!}. \tag{5.4}$$

If we define

$$f_N(K)=\theta(N-1-(m+n+r+s+t))v(k),\ k=(m,n,r,s,t)$$

where $\theta(x)=1(x\geqslant 0)$ and $0(x<0)$, we can explicitly write out the above finite-dimensional representation in the explicit form that is obtained by substituting $u[K]$ in (5.3) by $f_N(K)$. Its lowest non-trivial example is a six-dimensional representation

$$\begin{array}{lll} a=E_{1,6} & \bar{a}=-E_{5,1}+E_{2,6} & E=E_{5,6} \\ b=E_{3,6} & \bar{b}=E_{4,6} & H=\mu\sum_{i=1}^{6}E_{i,i}+E_{4,1}-E_{3,2} \end{array}\tag{5.5}$$

on an ordered basis

$$\begin{array}{lll} f_2(1,0,0,0,0) & f_2(0,1,0,0,0) & f_2(0,0,1,0,0) \\ f_2(0,0,0,1,0) & f_2(0,0,0,0,1) & f_2(0,0,0,0,0) \end{array}$$

where $E_{i,j}$ are the matrix units with the corresponding elements

$$(E_{i,j})_{r,s}=\delta_{i,r}\delta_{j,s}.$$

One purpose of building a quantum double is to obtain the solutions of the QYBE in terms of its universal R-matrix and matrix representations. In order to find the solutions of the QYBE associated with the exotic quantum double D, we have studied representation theory and construct both finite and infinite-dimensional representations of $\boldsymbol{D}$. In fact, for a given representation $T^{[x]}$ of $\boldsymbol{D}$:

$$T^{[x]}:\boldsymbol{D}\rightarrow \mathrm{End}(V)$$

on the linear space V where x is a continuous parameter, we can construct a R-matrix

$$R(x,y)=T^{[x]}\otimes T^{[y]}(\hat{R})$$

satisfying the QYBE

$$R_{1,2}(x,y)R_{1,3}(x,z)R_{2,3}(y,z)=R_{2,3}(y,z)R_{1,3}(x,z)R_{1,2}(x,y).\tag{5.6}$$

Here, x, y and z appear as the colour parameters similar to the non-additive spectrum parameters in QYBE. For example, using the above obtained six-dimensional representation, we can construct a 36×36- R-matrix

$$R=\exp(E_{1,6}\otimes E_{3,6})\exp([-E_{5,1}+E_{2,6}]\otimes E_{4,6})\exp\left(E_{5,6}\otimes\left(\mu\sum_{i=1}^{6}E_{i,i}+E_{4,1}-E_{3,2}\right)\right)$$

$$=(1+E_{1,6}\otimes E_{3,6})(1+[-E_{5,1}+E_{2,6}])(1+E_{5,6}\otimes\left(\mu\sum_{i=1}^{6}E_{i,i}+E_{4,1}-E_{3,2}\right).$$

It is pointed out that the higher-dimensional representations can also be obtained in the same form.

6. Generalization and discussion

To conclude this paper, we give some remarks on our exotic quantum double and its relations to the known results and try to extend them for more general Lie algebras.

From the construction of the exotic quantum double in this paper, we can see that a commutative (Abelian) algebra, e.g. the subalgebra $\boldsymbol{B}$, can be endowed with a non-commutative Hopf algebraic structure and its quantum dual $\boldsymbol{A}$ and quantum double $\boldsymbol{D}$ can be deduced as non-commutative algebras in an inverse process of the construction in this paper. Such a process possibly provide us with a scheme of 'quantization' from commutative object to non-commutative one. An example of this 'quantization' was given [5] recently.

It has to be pointed out that there are some difficulties in the further developments in constructing the general quantum double associated with arbitrary Lie algebra. When one take the subalgebra B to be the whole UEA of an arbitrary Lie algebra, we hardly write down the dual basis explicitly and so the construction scheme of this paper can not work well. However, the method in this paper is applicable to another nilpotent Lie algebra $A(n)$, the universal enveloping algebra of the $(n+1)$-dimensional HW-Lie algebra, whose basis is a_i, $\bar{a}_i$ $(i=1, 2, \ldots, n)$ and E with commutation relations

$$[a_i, \bar{a}_i]=E \qquad [E, a_i]=0=[E, \bar{a}_i]. \tag{6.1}$$

Let $b_i, \bar{b}_i$ and H be the dual generators for the dual algebra $B(n)$ to a_i, $\bar{a}_i$ and H respectively. Then, the method in the last sections leads to a quantum double $D(n)$ with the Hopf algebra structure

$$\begin{aligned}
&\Delta(x)=x\otimes 1+1\otimes x \qquad S(x)=-x\\
&\varepsilon(x)=0 \qquad x\in A(n) \qquad \text{or} \qquad x=b_i, \bar{b}_i\\
&\Delta(H)=H\otimes 1+1\otimes H+\sum_{i=1}^{n} \bar{b}_i\otimes b_i
\end{aligned} \tag{6.2}$$

and the only non-zero commutators as the multiplication relations

$$[a_i, \bar{a}_i]=E \qquad [H, a_i]=\bar{b}_i \qquad [H, \bar{a}_i]=-b_i. \tag{6.3}$$

The following intertwiner, the universal R-matrix,

$$\hat{R}=\prod_{i=1}^{n} \exp(\bar{a}_i\otimes\bar{b}_i) \exp(a_i\otimes b_i) \exp(E\otimes H) \tag{6.4}$$

enjoys the quasi-triangular structure. It has to be pointed out that how to apply this method to any nilpotent Lie algebra is still an open question.

In the formal group theory of Lie algebra [8], the bialgebra structure of the dual to the UEA of a classical Lie algebra can be given abstractly in terms of the formal group. It is not difficult to further define the antipode for this dual bialgebra. So, in this abstract way, the Hopf algebraic structure can be endowed with the dual Hopf algebra of the UEA. However, writing out the explicit Hopf algebraic structure, namely, the explicit multiplication relations, coproduct, antipode and co-unit for the dual generators, completely depends on the explicit evolution of the Baker–Comppell–Hausdorff formula for classical Lie algebra, and it is difficult to do this even for a simple case, e.g. $SU(2)$. The study in this paper avoids this evaluation so that not only the dual Hopf algebraic structure is obtained, but also the corresponding quantum double—the exotic quantum double is built for the Borel subalgebra of the UEA of arbitrary classical Lie algebra by combining the two subalgebras dual to each other.

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